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An Iterative Algorithm for the Numerical Solution of Singular Integral Equations

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Several problems of mathematical physics are reducible to singular integral equations of the first kind with Cauchy-type kernels. In this paper an iterative method for the numerical solution of this class of equations is proposed. This method can be considered analogous to the existing iterative methods for the numerical solution of Fredholm integral equations of the second kind and is based on the Gauss– and Lobatto–Chebyshev quadrature rules. The method is applied to some plane elasticity crack problems and is seen to give convergent results. The application of the proposed method to other singular integral equations appearing in physical and engineering problems, either without modifications or after appropriate modifications, is trivial.

1. INTRODUCTION

Several engineering and physical problems of practical interest are reducible to singular integral equations (with Cauchy-type kernels). Moreover, several numerical methods for the approximate solution of singular integral equations have appeared during the last decade. Some of these methods are reported in the literature of [1]. Yet, none of these methods is iterative, as happens with the successive approximations method (including the Neumann iterations method) [2, 3] and other more sophisticated iterative methods described in [2].

In this paper we describe a simple iterative method for the numerical solution of singular integral equations. This method is based on the Gauss– and Lobatto–Chebyshev numerical integration rules for regular [4] and Cauchy-type principal value integrals [5–7]. The solutions are achieved without solving systems of linear algebraic equations.

Our iterative algorithm will be applied to crack problems in plane or antiplane elasticity (see, e.g., [5] and the literature mentioned in [1, 5, 8]). These equations are of the first kind and with index κ [9] equal to 1 and have the form

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\frac{g(t)}{t-x}\,dt+\frac{1}{\pi}\int_{-1}^{1} w(t)\,k(t,x)\,g(t)\,dt=f(x),\qquad -1<x<1, \quad (1.1)$$

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where $w(t)$ is the weight function,

$$w(t) = (1 - t^2)^{-1/2}, \tag{1.2}$$

$g(t)$ is the unknown function (proportional to the dislocations density along the crack [5]), $k(t, x)$ is a regular kernel and $f(x)$ is a known regular function. Since $\kappa = 1$, (1.1) is supplemented by the condition

$$\int_{-1}^{1} w(t)\, g(t)\, dt = 0. \tag{1.3}$$

This is the condition of single-valuedness of displacements in crack problems and assures the uniqueness of the solution $g(t)$ of (1.1).

During the last decade more than 100 papers dealing with the numerical solution of (1.1) and (1.3), under various geometry and loading conditions, have appeared. Collocation and quadrature methods have mainly been used to reduce (1.1) and (1.3) to a system of linear algebraic equations. (First proposed by Kalandiya [10, 11], they were further generalized by Erdogan and Gupta [12], and, via a "natural" interpolation formula [2] and the Lobatto–Chebyshev quadrature rule, greatly improved by Ioakimidis and co-workers [5, 13, 14].) Some theoretical results justifying the use of the above methods were found by Ioakimidis and reported in [15–18].

It is believed that the iterative alorithm for the numerical solution of (1.1) and (1.3) will also prove useful in practice for the solution of these equations and the determination of stress-intensity factors, proportional to the values $g(\pm 1)$ at the endpoints $t = \pm 1$ of the integration interval $[-1, 1]$. The main advantage of the proposed method is that it has no need of the numerical solution of systems of linear algebraic equations and, moreover, it is very simple to program. To date no theoretical proof of the convergence of the proposed algorithm is available. Results calculated via this algorithm converge in three cases of plane elasticity crack problems.

The numerical techniques developed in order to obtain numerical solutions to Fredholm integral equations of the second kind [2, 3] are basic to the new algorithm for singular integral equations. Consider the Fredholm integral equation of the second kind [2, 3]

$$G(x) + \int_{-1}^{1} w(y)\, K(x, y)\, G(y)\, dy = F(x), \qquad -1 \leqslant x \leqslant 1, \tag{1.4}$$

where $w(y)$ is defined by (1.2), $K(x, y)$ and $F(x)$ are known functions and $G(x)$ is the unknown function. One simple way for solving this equation numerically is to use an iterative scheme (successive approximations or Neumann iterations) [2, 3]

$$G_{k+1}(x) = F(x) - \int_{-1}^{1} w(y)\, K(x, y)\, G_k(y)\, dy, \qquad k = 0, 1, 2, \ldots, \tag{1.5}$$

with

$$G_0(x) = F(x). \tag{1.6}$$

(In these equations $G_k(x)$ denote approximations to the unknown function $G(x)$.) This algorithm is convergent under well-known conditions that are generally fulfilled. In practice we are seldom able to find a closed-form expression for the integral in (1.5) and we have to use a quadrature rule of the form [2, 4]

$$\int_{-1}^{1} w(y)\, H(y)\, dy \simeq \sum_{i=1}^{n} A_{i,n} H(y_{i,n}), \tag{1.7}$$

where $y_{i,n}$ are the nodes and $A_{i,n}$ the weights. Then (1.5) takes the form [2]

$$G_{k+1}(x) = F(x) - \sum_{i=1}^{n} A_{i,n} K(x, y_{i,n})\, G_k(y_{i,n}). \tag{1.8}$$

Although the above algorithm is a satisfactory one, an improvement and generalization of it was also suggested. Rewrite (1.4) as

$$G(x) + \int_{-1}^{1} w(y)\, K(x,y)\, G(y)\, dy = F(x) - \int_{-1}^{1} w(y)\, K(x,y)\, G(y)\, dy + \int_{-1}^{1} w(y)\, K(x,y)\, G(y)\, dy, \qquad -1 \leqslant x \leqslant 1, \tag{1.9}$$

and (1.5) as

$$G_{k+1}(x) + \int_{-1}^{1} w(y)\, K(x,y)\, G_{k+1}(y)\, dy \simeq F(x) - \int_{-1}^{1} w(y)\, K(x,y)\, G_k(y)\, dy + \int_{-1}^{1} w(y)\, K(x,y)\, G_k(y)\, dy. \tag{1.10}$$

Next, apply the quadrature rule (1.7) to the evaluation of the first integral of the right side of (1.10) and the same rule, but with m nodes ($m < n$), to the evaluation of the remaining two integrals of (1.10). Then we obtain

$$G_{k+1}(x) + \sum_{i=1}^{m} A_{i,m} K(x, y_{i,m})\, G_{k+1}(y_{i,m}) = F(x) - \sum_{i=1}^{n} A_{i,n} K(x, y_{i,n})\, G_k(y_{i,n}) + \sum_{i=1}^{m} A_{i,m} K(x, y_{i,m})\, G_k(y_{i,m}). \tag{1.11}$$

By putting $x = y_{i,m}$ in (1.11), we obtain a system of m linear algebraic equations for the determination of the values of $G_{k+1}(y_{i,m})$, the function $G_k(x)$ assumed known. Of course, an original assumption for $G(x)$, like (1.6), has to be made. After the deter-

mination of $G_{k+1}(y_{i,m})$, (1.11) serves as a natural interpolation formula for the estimation of $G_{k+1}(x)$ along the whole interval $[-1, 1]$.

The above-described algorithm for the numerical solution of Fredholm integral equations of the second kind is well known and completely investigated. Atkinson [2] refers to it as the first iteration method in the section of iterative variants of the Nyström method. In [2] this algorithm is not only described but also proved convergent. Moreover, it is clear that if we select $m = 0$ in (1.11), then this equation reduces to (1.8). Hence, the Neumann iterations (1.5), or (1.8), are essentially a special case of the aforementioned iterative algorithm of Atkinson.

It was the purpose of this paper to generalize the Neumann iterations (1.5), or (1.8), directly to the case of singular integral equations of the form (1.1) (together with the supplementary condition (1.3)). Unfortunately, this was found impossible since Eq. (1.1) contains the unknown function $g(t)$ only under the integral sign and, therefore, no formula for the estimation of $g(t)$ along the whole interval $[-1, 1]$ could be found. At this point, we took into account the results of [14–18], where it was suggested that a natural interpolation formula for singular integral equations can be derived from the error term of a quadrature rule for Cauchy-type principal value integrals. This was formally justified in [18], where it was proved that such a natural interpolation formula is equivalent to the classical natural interpolation formula of Nyström for Fredholm integral equations of the second kind [2, 3].

Under these conditions, we are not able to use (1.11) with $m = 0$ (that is, the Neumann iterations, (1.5) and (1.8)). We need a quadrature rule for the derivation of our natural interpolation formula (like (1.11) after the determination of $G_{k+1}(y_{i,m})$) and for this reason we chose the value $m = 1$. In this way, the algorithm to be described in the next section (as a straightforward generalization of the aforementioned results for Fredholm integral equations of the second kind) is sufficiently simple, since the selection $m = 1$ eliminates the task of solving systems of linear algebraic equations (as could be necessary in (1.11) for $m > 1$). It is possible to generalize the results of the following section (by selecting $m > 1$), but then the algorithm becomes more complicated.

We wish to mention again that the algorithm to be suggested in the next section is the simplest possible and the most relevant to Neumann iterations, which seem not directly applicable to singular integral equations. Moreover, it is a straightforward generalization of Atkinson's method for Fredholm integral equations of the second kind to the case of singular integral equations.

2. The Iterative Algorithm

By taking into account the developments of the previous section, i.e., the successive approximations (Neumann iterations) method and the iterative methods for the numerical solution of Fredholm integral equations of the second kind by using numerical integration rules [2], as well as the method of estimating the unknown function in a singular integral equation from the error term of the numerical

integration rule used [14], we can suggest the following iterative algorithm for the numerical solution of the singular integral equation (1.1) accompanied by the condition (1.3).

First, consider the Gauss–Chebyshev and the Lobatto–Chebyshev quadrature rules for the numerical evaluation of regular integrals with the weight function $w(t)$, defined by Eq. (1.2) [4],

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\,\varphi(t)\,dt \simeq \frac{1}{m}\sum_{i=1}^{m} \varphi(t_i), \tag{2.1}$$

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\,\varphi(t)\,dt \simeq \frac{1}{m}\sum_{j=0}^{m}{}'' \varphi(u_j), \tag{2.2}$$

respectively, where the primes in the sum of (2.2) mean that the terms corresponding to $j=0$ and $j=m$ should be halved. The nodes t_i and u_j are given by

$$t_i = \cos[(i-0.5)\,\pi/m], \qquad i = 1(1)m, \tag{2.3}$$

$$u_j = \cos(j\pi/m), \qquad j = 0(1)m. \tag{2.4}$$

Assume that the number of nodes m is sufficiently large so that the error terms neglected are insignificant when the integrals in (1.1) and (1.3) are evaluated numerically.

Moreover, since the first integral in (1.1) is a Cauchy-type principal value integral, we take also into account the modified forms of (2.1) and (2.2) for such integrals [5, 6]

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\frac{\varphi(t)}{t-x}\,dt \simeq \frac{1}{m}\sum_{i=1}^{m}\frac{\varphi(t_i)}{t_i-x} + \frac{U_{m-1}(x)}{T_m(x)}\varphi(x), \qquad x \neq t_i,\ i = 1(1)m, \tag{2.5}$$

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\frac{\varphi(t)}{t-x}\,dt \simeq \frac{1}{m}\sum_{j=0}^{m}{}''\frac{\varphi(u_j)}{u_j-x} - \frac{T_m(x)}{(1-x^2)\,U_{m-1}(x)}\varphi(x), \qquad x \neq u_j,\ j = 0(1)m, \tag{2.6}$$

where $T_m(x)$ and $U_m(x)$ denote the Chebyshev polynomials of the first and the second kinds, respectively, and degree m. The first of these quadrature rules is the Gauss–Chebyshev rule and the second is the Lobatto–Chebyshev rule for Cauchy-type principal value integrals.

Now we make an initial assumption $g_0(t)$ for the unknown function $g(t)$ in (1.1) and (1.3) (for example, by assuming that $k(t,x) \equiv 0$, whence (1.1) and (1.3) possess a closed-form solution [9]) and derive the sequence of approximations $g_k(t)$ to $g(t)$ as follows: As regards condition (1.3), rewrite it as

$$\int_{-1}^{1} w(t)\,g_{k+1}(t)\,dt - \int_{-1}^{1} w(t)\,g_k(t)\,dt \simeq -\int_{-1}^{1} w(t)\,g_k(t)\,dt. \tag{2.7}$$

Next, apply (2.1) with $m = 1$ to the approximation of the first two integrals in this equation and with $m = n$ to the approximation of the third integral. Then we obtain

$$g_{k+1}(0) = g_k(0) - \frac{1}{n} \sum_{i=1}^{n} g_k(t_i). \tag{2.8}$$

This is the first equation which will be used in our algorithm. Evidently, if $g_{k+1}(0) - g_k(0) \to 0$ as $k \to \infty$, then

$$\frac{1}{n} \sum_{i=1}^{n} g_\infty(t_i) = 0 \tag{2.9}$$

and (1.3) will be satisfied if the quadrature rule (2.1) with n nodes is a sufficiently accurate rule for the unknown function $g(t)$, as was already assumed.

Similarly, rewrite (1.1) as

$$\begin{aligned} &\frac{1}{\pi} \int_{-1}^{1} w(t)\, K(t, x)\, g_{k+1}(t)\, dt - \frac{1}{\pi} \int_{-1}^{1} w(t)\, K(t, x)\, g_k(t)\, dt \\ &\quad \simeq f(x) - \frac{1}{\pi} \int_{-1}^{1} w(t)\, K(t, x)\, g_k(t)\, dt, \qquad -1 < x < 1, \end{aligned} \tag{2.10}$$

where

$$K(t, x) = 1/(t - x) + k(t, x). \tag{2.11}$$

Apply (2.1) and (2.5) with $m = 1$ to the approximation of the first two integrals in this equation and with $m = n$ to the approximation of the third integral. Then we have

$$\begin{aligned} g_{k+1}(x) = g_k(x) + [1 - xk(0, x)][g_{k+1}(0) - g_k(0)] + x[f(x) \\ - \frac{1}{n} \sum_{i=1}^{n} K(t_i, x)\, g_k(t_i) - U_{n-1}(x) g_k(x)/T_n(x)], \\ x \neq t_i,\ i = 1(1)n. \end{aligned} \tag{2.12}$$

Apply (2.12) at the collocation points u_j defined by (2.4), which are the roots of the polynomial $(1 - x^2)\, U_{n-1}(x)$. We obtain

$$\begin{aligned} g_{k+1}(u_j) = (1 - n\delta_j)\, g_k(u_j) + [1 - u_j k(0, u_j)][g_{k+1}(0) - g_k(0)] \\ + u_j \left[f(u_j) - \frac{1}{n} \sum_{i=1}^{n} K(t_i, u_j) g_k(t_i) \right], \qquad j = 0(1)n, \end{aligned} \tag{2.13}$$

where

$$\delta_j = 0 \text{ if } j = 1(1)(n-1), \qquad \delta_j = 1 \text{ if } j = 0 \text{ or } j = n. \tag{2.14}$$

Unfortunately, (2.8) and (2.13) are not sufficient for the construction of the algorithm since both the values of $g_k(u_j)$ and $g_k(t_i)$ are involved in the second of them. Hence, we use (2.10) once more but with the Lobatto–Chebyshev quadrature rules (2.2) and (2.6) applied to the approximation of the third integral in (2.10). Then we obtain

$$g_{k+1}(x) = g_k(x) + [1 - xk(0, x)][g_{k+1}(0) - g_k(0)] + x\left\{f(x) - \frac{1}{n}\sum_{j=0}^{n}{}'' K(u_j, x) g_k(u_j) + T_n(x) g(x)/[(1-x^2)\, U_{n-1}(x)]\right\}, \quad x \neq u_j, j = 0(1)n. \tag{2.15}$$

At the collocation points t_i, defined by (2.3) (with $m = n$), which are the roots of the Chebyshev polynomial $T_n(x)$, (2.15) becomes

$$g_{k+1}(t_i) = g_k(t_i) + [1 - t_i k(0, t_i)][g_{k+1}(0) - g_k(0)] + t_i\left[f(t_i) - \frac{1}{n}\sum_{j=0}^{n}{}'' K(u_j, t_i)\, g_k(u_j)\right], \quad i = 1(1)n. \tag{2.16}$$

Equations (2.8), (2.13) and (2.16) describe completely the iterative algorithm proposed in this section. Provided that the values $g_k(t_i)$ and $g_k(u_j)$ are known (clearly the node $t = 0$ is always included among the nodes t_i and u_j), then the value $g_{k+1}(0)$ is determined from (2.8), the values $g_{k+1}(t_i)$ are determined from (2.16) and the values $g_{k+1}(u_j)$ are determined from (2.13). These values can then be used for the next iteration. Equation (2.12) or (2.15) can be used for the estimation of $g_{k+1}(x)$ along the whole interval $[-1, 1]$, if required, although this determination is not necessary during the iterations procedure.

If, as $k \to \infty$, $g_{k+1}(t_i) - g_k(t_i) \to 0$ and $g_{k+1}(u_j) - g_k(u_j) \to 0$, then (2.8), (2.13) and (2.16) yield (2.9), as well as the following linear algebraic equations:

$$\frac{1}{n}\sum_{i=1}^{n} K(t_i, u_j)\, g_\infty(t_i) = f(u_j), \quad j = 0(1)\, n, \tag{2.17}$$

$$\frac{1}{n}\sum_{j=0}^{n}{}'' K(u_j, t_i) g_\infty(u_j) = f(t_i), \quad i = 1(1)\, n. \tag{2.18}$$

Clearly, Eqs. (2.17) are the linear algebraic equations resulting when the Gauss–Chebyshev direct quadrature method [5, 12, 13] is applied to the numerical solution of (1.1) and (1.3). Similarly, (2.18) are the corresponding equations when the Lobatto–Chebyshev direct quadrature method [5, 13] is used for the numerical solution of the same equations. Thus, the values $g_\infty(t_i)$ coincide numerically ((2.9) taken into account) with the corresponding values resulting from the Gauss–Chebyshev method by also using n nodes. But here the number n may be very large (since no system of linear algebraic equations has to be solved during the

present iterative algorithm) so that the quadrature rules used to derive (2.9), as well as (2.17) and (2.18), are nearly exact. On the other hand, always in the case of convergence, the values $g_\infty(u_j)$ are slightly different from the corresponding numerical values derived in the aforementioned Lobatto–Chebyshev direct quadrature method of numerical solution of (1.1) and (1.3). This is due to the fact that the Gauss–Cheebyshev quadrature rule (2.1) has been used in (2.8) and not the Lobatto–Chebyshev quadrature rule. It is possible to replace (2.8) in the algorithm by the equation

$$g_{k+1}(0) = g_k(0) - \frac{1}{n} \sum_{j=0}^{n}{}'' g_k(u_j) \tag{2.19}$$

by taking into account (2.2). The accuracy of the numerical results obtained will be essentially the same as previously since both quadrature rules (2.1) and (2.2) are exact for integrands $\varphi(t)$, polynomials of up to $(2n-1)$ degree.

After the determination of the values $g_\infty(t_i)$ and $g_\infty(u_j)$, $g_\infty(x)$ can be determined along the whole interval $[-1, 1]$ as the Lagrangian interpolation polynomial based on these values or by the interpolation methods suggested in [14]. In the case of crack problems the values of the reduced stress-intensity factors $k^*(\pm 1)$ are determined by [5]

$$k^*(\pm 1) = \pm g_\infty(\pm 1). \tag{2.20}$$

These values are directly available, without using interpolation–extrapolation techniques since $u_0 = 1$ and $u_n = -1$ as is clear from Eqs. (2.4) (with $m = n$).

3. Applications to Crack Problems

In this section we apply the iterative algorithm of the previous section to some simple plane isotropic elasticity crack problems. In all these problems the algorithm provides convergent results as $k \to \infty$.

As a first application consider the problem of a simple straight crack inside a plane isotropic elastic medium (see, e.g., [5]). In this case

$$k(t, x) \equiv 0, \qquad K(t, x) = 1/(t - x). \tag{3.1}$$

Assume that the loading (pressure) distribution $f(x)$ along both crack edges is given by

$$f(x) = \exp x \tag{3.2}$$

and solve (1.1) and (1.3) by the algorithm described above. Make the following initial guess for $g(t)$,

$$g_0(t) = t, \tag{3.3}$$

corresponding to the exact solution of our problem for

$$f(x) = 1. \tag{3.4}$$

Then from (2.20) we find, because of (3.3), that

$$k_0^*(\pm 1) = 1 \tag{3.5}$$

for the initial estimations of the stress-intensity factors at the crack tips. The correct values for these factors are [14]

$$k^*(1) = 1.8312, \qquad k^*(-1) = 0.7009. \tag{3.6}$$

As is clear from a comparison of (3.2) and (3.4), as well as (3.5) and (3.6), the initial assumption for $g(t)$ was far from good. Yet, the numerical results obtained by applying the algorithm of the previous section converged.

Table I (second and third columns) shows the values of $k_k^*(\pm 1)$ determined by

$$k_k^*(\pm 1) = \pm g_k(\pm 1) \tag{3.7}$$

after the application of the present algorithm to (1.1) and (1.3) under the present conditions with $n = 4$ and $k = 0(1)6$, as well as $k \to \infty$ (obtained in practice for $k = 10$). From these results we observe the rapid convergence of the numerical results. Next, Table II (second and third columns) shows the values of $k_\infty^*(\pm 1)$ (determined also with $k = 10$) for $n = 2(1)5$. From these results it is clear that in the present application n need not be large to obtain accurate results for $k^*(\pm 1)$ (3.6).

TABLE I

Convergence of the Numerical Results for the Stress-Intensity Factors $k^*(\pm 1)$ at the Tips of a Simple Straight Crack or a Periodic Array of Collinear or Parallel Cracks (with $a/b = 0.25$) for Two Loading Distributions $f(x)$ for Increasing Values of k and $n = 4$

	One crack			Collinear cracks				Parallel cracks
$f(x)$:	$\exp x$	$\exp x$	$x^2\lvert x\rvert$	1	$\exp x$	$\exp x$	$x^2\lvert x\rvert$	1
a/b:	0	0	0	0.25	0.25	0.25	0.25	0.25
k	$k_k^*(1)$	$k_k^*(-1)$	$k_k^*(\pm 1)$	$k_k^*(\pm 1)$	$k_k^*(1)$	$k_k^*(-1)$	$k_k^*(\pm 1)$	$k_k^*(\pm 1)$
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	2.7183	0.3679	1.0000	1.1228	2.8411	0.4907	1.1228	0.7531
2	1.7741	0.7358	0.3029	1.1283	1.9372	0.9286	0.3926	0.7953
3	1.8325	0.6999	0.4042	1.1283	1.9829	0.8406	0.4119	0.7887
4	1.8312	0.7010	0.4223	1.1284	1.9729	0.8497	0.4426	0.7898
5	1.8312	0.7009	0.4223	1.1284	1.9745	0.8483	0.4436	0.7896
6	1.8312	0.7009	0.4223	1.1284	1.9743	0.8486	0.4438	0.7896
∞	1.8312	0.7009	0.4223	1.1284	1.9743	0.8485	0.4437	0.7896

TABLE II

Results as in Table I but for Increasing Values of n

	One crack			Collinear cracks				Parallel cracks
$f(x)$:	$\exp x$	$\exp x$	$x^2\|x\|$	1	$\exp x$	$\exp x$	$x^2\|x\|$	1
a/b:	0	0	0	0.25	0.25	0.25	0.25	0.25
n	$k^*_\infty(1)$	$k^*_\infty(-1)$	$k^*_\infty(\pm 1)$	$k^*_\infty(\pm 1)$	$k^*_\infty(1)$	$k^*_\infty(-1)$	$k^*_\infty(\pm 1)$	$k^*_\infty(\pm 1)$
2	1.8482	0.6730	0.3536	1.1307	2.0129	0.8377	0.3998	0.7900
3	1.8315	0.7007	0.4167	1.1283	1.9741	0.8475	0.4327	0.7896
4	1.8312	0.7009	0.4223	1.1284	1.9743	0.8485	0.4437	0.7896
5	1.8312	0.7009	0.4236	1.1284	1.9743	0.8485	0.4455	0.7896

Even if this were the case, the direct application of our algorithm would still be possible. In practice, it is convenient to use a sufficiently large value for n (estimated from the smoothness of the known functions $k(t, x)$ and $f(x)$ in (1.1)) in the algorithm so that no repetition of it with higher values of n is necessary or just one repetition with n replaced, for example, by $2n$ to confirm the original numerical results.

We have also considered the problem of a periodic array of cracks of length $2a$ along a straight line with a period equal to b also inside a plane isotropic elastic medium. In this application the kernels $k(t, x)$ and $K(t, x)$, related by (2.11), are given by [5, 19]

$$K(t, x) = \frac{\pi a}{b} \cot \frac{\pi a(t-x)}{b}, \qquad k(t, x) = K(t, x) - \frac{1}{t-x}. \tag{3.8}$$

Numerical results, similar to those described previously in the case of a simple straight crack (with $b = \infty$, that is $a/b = 0$), are also presented in Tables I and II for $a/b = 0.25$ and a loading distribution $f(x)$ along the crack edges either constant ((3.4); fifth columns), or exponential ((3.2); sixth and seventh columns). The numerical results were also seen to converge rapidly in spite of the fact that (3.3) was assumed again as an initial approximation to $g(t)$. For $f(x) = 1$, the theoretical value for the stress-intensity factor $k^*(\pm 1)$ at the crack tips is determined by (see, e.g., [5])

$$k^*(\pm 1) = \left(\frac{\pi a}{b} \cot \frac{\pi a}{b} \right)^{-1/2}. \tag{3.9}$$

From this formula we obtain the value $k^*(\pm 1) = 1.1284$ for $a/b = 0.25$, which is in complete agreement with the numerical results of Tables I and II for this case. The results of Table I show the rapid convergence of the numerical results obtained by the algorithm, even for very bad original assumptions for $g_0(t)$.

Next, the algorithm with the same initial guess for $g_0(t)$ was applied to the case

when $a/b = 0.4$. In this case, each crack is four times longer than the distance between two successive crack tips of two successive cracks. In spite of this inconvenient geometry condition, the algorithm produced convergent results for both selections (3.4) and (3.2) for the loading distribution $f(x)$. The convergence, while not as rapid as in the numerical results of Tables I and II, was acceptable.

To further test the algorithm, consider the case of a periodic array of parallel cracks, the length of the cracks denoted again by $2a$ and the period of the array by b. In this case, (3.8) take the forms [5, 19]

$$K(t,x) = \frac{\pi a}{b}\left[2\coth\frac{\pi a(t-x)}{b} - \frac{\pi a(t-x)}{b}\operatorname{cosech}^2\frac{\pi a(t-x)}{b}\right],$$

$$k(t,x) = K(t,x) - \frac{1}{t-x}. \tag{3.10}$$

Numerical results for this problem, with $a/b = 0.25$ and $f(x) = 1$, are presented in Tables I and II (ninth columns) and the convergence of the algorithm is seen from Table I (for increasing values of k). The convergence of the corresponding results of Table II denotes essentially the convergence of the Gauss– and Lobatto–Chebyshev methods for singular integral equations and not of the present algorithm. Furthermore, the value $k^* = 0.7896$ in the present application (clear from the results of Tables I and II) is in complete agreement with the corresponding value reported in [5, 19].

The kernels $K(t,x)$ used in the previous applications were sufficiently complicated to illustrate the power of the proposed algorithm. The right-hand side functions $f(x)$ in the examples above are well-behaved. Now consider the case where

$$f(x) = x^2|x| \tag{3.11}$$

with only two continuous derivatives. The results of [17] demonstrate that continuity of only the first derivative of $f(x)$ is not sufficient to guarantee the convergence of the Gauss– and Lobatto–Chebyshev methods. The continuity of the second derivative of this function seems required. Hence, $f(x)$, as defined by (3.11), having just the minimum properties for the convergence of these methods, was selected as a test function in [17].

The new algorithm, operating on the loading distribution (3.11) with either a single straight crack or a periodic array of collinear cracks (with $a/b = 0.25$), produced the numerical results for the stress-intensity factors displayed in Tables I and II (fourth and eighth columns). Comparison with the corresponding numerical results reported in [17] is excellent. The bad behavior of $f(x)$ in the present application had negligible influence on the speed of convergence (Table I, increasing values of k). From the results of Table II it is clear that larger values of n (compared with those used in the previous applications) must be used to obtain the same accuracy in the final numerical results.

4. Generalizations

The algorithm proposed in Section 2 can be appropriately modified in order to solve related physical problems, for example, fluid mechanics problems, contact problems, dislocation problems, and waveguide problems.

Replacing the Gauss– and Lobatto–Chebyshev quadrature rules by the Gauss– and Lobatto–Jacobi quadrature rules creates a new algorithm, applicable to singular integral equations of the second kind (with index also equal to 1). Examples considered by the direct quadrature method in [20] for the case of real singularities α and β in the weight function $w^*(t)$ associated with the Jacobi polynomials

$$w^*(t) = (1-t)^\alpha (1+t)^\beta \tag{4.1}$$

or in [21] for the case of complex singularities α and β are readily solved.

Another possibility substitutes two semi-closed Radau-type quadrature rules (one with a node coinciding with the end-point $t = 1$ of the integration interval $[-1, 1]$ and the other with a node coinciding with the other end-point $t = -1$ of the same interval) [20] instead of the corresponding Gauss-type and Lobatto-type rules in both cases of singular integral equations of the first or the second kind (associated with the Chebyshev or the Jacobi polynomials, respectively).

Moreover, the new iterative algorithm can solve systems of uncoupled singular integral equations of the form (1.1) (or systems of singular integral equations of the second kind).

In fluid mechanics problems or contact problems in plane elasticity one of the restrictions

$$g(1) = 0 \qquad \text{or} \qquad g(-1) = 0 \tag{4.2}$$

for the unknown function $g(t)$ in (1.1) often applies. In this case the index κ of (1.1) is equal to 0 and condition (1.3) is no longer necessary and valid. Then (2.7), (2.8) and (2.9) are not applicable. In order to use the algorithm of Section 2 in this case, construct a condition similar to Eq. (1.3) by applying Eq. (1.1) at the end-point $t = 1$ or $t = -1$ of the interval $[-1, 1]$ where $g(t)$ vanishes. Clearly, Eq. (1.1) is valid for this end-point of the interval $[-1, 1]$, but, obviously, it is not valid for the other end-point of the same interval (since the principal value Cauchy-type integral of Eq. (1.1) is not defined for the latter end-point). If, for example, $g(1) = 0$, we obtain, from Eq. (1.1) for $x = 1$,

$$\frac{1}{\pi} \int_{-1}^{1} w(t) K(t, 1) g(t)\, dt = f(1), \tag{4.3}$$

where $K(t, x)$ is defined again by (2.11). This equation can be used exactly as (1.3) for the derivation of equations similar to (2.7) to (2.9).

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